THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5210 Discrete Mathematics 2017-2018
Supplementary Exercises on Graphs

1. Let G be a simple graph, then its complement $\overline{G}$ is defined as the simple graph with the same vertex set as G in which two vertices are adjacent if and only if they are not adjacent in G.

   (a) What is the complement of the complete graph K_n? What is the complement of the complete bipartite graph $K_{r,s}$?

   (b) Prove that a simple graph and its complement cannot both be disconnected.

   Ans:

   (a) The complement of K_n is the null graph N_n and the complement of $K_{r,s}$ is a union of two complete graphs K_r and K_s.

   (b) Let G be disconnected. Then we claim that $\overline{G}$ is connected.

   Let v and w be two vertices of G. If they lie in different component of G, then they are adjacent in $\overline{G}$; otherwise, let z be a vertex such that z lies in another complement of G, then $v \to z \to w$ is a path in $\overline{G}$. In either case, any two vertices can be connected by a path in $\overline{G}$, and hence $\overline{G}$ is connected.

2. (a) Prove that G is a bipartite graph if and only if each cycle of G has even length.

   (b) Prove that every tree is a bipartite graph.

   Ans:

   (a)
   - If G is a bipartite graph, we can split its vertex set into two disjoint set A and B so that each edge of G joins a vertex of A and a vertex of B. Let $v_0 \to v_1 \to v_2 \to \cdots \to v_n \to v_0$ be a cycle in G. Assume v_0 is in A, then v_1 must be in B, v_2 must be in A and so on. Since v_n must be in B, the cycle has even length.
   - We may assume G is connected, otherwise, we can carry out the procedure below on each component.

     Suppose that each cycle of G has even length. Choose any vertex v in G. Divide G into two sets of vertices like this:

     Let A be the set of vertices such that the shortest path from each element of A to v is of odd length and Let B be the set of vertices such that the shortest path from each element of B to v is of even length.

     Then, $v \in B$ and $A \cap B$ is empty.

     We claim that every pair of vertices in A is not adjacent. Otherwise, there exists $v_1, v_2 \in A$ such that they are connected by an edge e. Then, consider the cycle containing the path from v to v_1, the edge e and the path from v_2 to v, it would be an odd cycle and it contradicts to the assumption. Similar argument can show that every pair of vertices in B is not adjacent. Therefore, G is a bipartite graph.

   (b) It is trivial, since a tree contains no cycles.

3. (a) For which values of n is the complete graph K_n Eulerian?

   (b) For which values of r and s is the complere bipartite $K_{r,s}$ Eulerian?

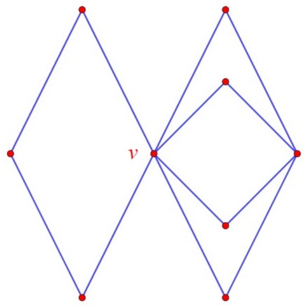
Ans:

(a) Since every vertex is of degree $n-1$ in K_n, K_n is Eulerian if and only if $n-1$ is even, i.e. n is odd.

(b) Since every vertex is either of degree r or s in $K_{r,s}$, $K_{r,s}$ is Eulerian if and only if r and s are even.

4. An Eulerian graph is **randomly traceable** from a vertex v if, whenever we start from v and transverse the graph in an arbitrary way never using any edge twice, we eventually obtain an Eulerian trail.

   (a) Show that the following graph is randomly traceable from v.



   (b) Give an example of Eulerian graph that is not randomly traceable.

(Remark: A randomly traceable graph may be used for the layout of an exhibition.)

Ans:

(a) Direct exhaustion.

(b) Change the vertex v to be the leftmost one.

(Remark: Furthermore, you may try to prove an Eulerian graph is randomly traceable from a vertex v if and only if every cycle of G contains v.)

5. (a) Prove that, if G is a bipartite graph with an odd number of vertices, then G is non-Hamiltonian.

   (b) Show that, if n is odd, it is not possible for a knight to visit all the squares of an $n \times n$ chessboard exactly once by knight's moves and return to its starting point.

Ans:

(a) Suppose the contrary. Let $v_0 \to v_1 \to v_2 \to \cdots \to v_n \to v_0$ be a Hamiltonian cycle in G. Since G is a bipartite graph, we can split its vertex set into two disjoint set A and B so that each edge of G joins a vertex of A and a vertex of B. Assume v_0 is in A, then v_1 must be in B, v_2 must be in A and so on. Then since n is odd, both v_0 and v_n are in A and they are adjacent, which leads a contradiction.

(b) We construct a graph G by regarding the squares on the chessboard as vertices and if the knight can move from a square to another, there is an edge joining the corresponding edges. Clearly, the resulting graph is a bipartite graph which n^2 vertices.

Note that if n is odd, n^2 is odd as well. By (a), the graph G is non Hamiltonian, so it is not possible for a knight to visit all the squares of the chessboard exactly once by knight's moves and return to its starting point.

6. Three unfriendly neighbors go to the same school, post office, and market. In order to avoid meeting, they wish to build non-crossing paths from each of their houses to each of the three buildings. Can this be done?

Ans: No, since $K_{3,3}$ is a non-planar graph. (Therefore, always have a good relation with your neighbors!)

7. (a) Use the Euler's formula to show that, if G is a connected planar graph such that the length of each cycle is at least 5, then $3E \leq 5V - 10$ where V and E are the number of vertices and edges respectively.

(b) Hence, deduce that the Petersen graph is non-planar.

Ans:

(a) Assume that we have a plane drawing of G. Since each face is bounded by at least 5 edges, it follows on counting up the egdes around each face that $5F \leq 2E$, where F is the number of faces. By Euler's formula, we have $F = 2 - V + E$. Therefore, $5(2 - V + E) \leq 2E$ which implies $3E \leq 5V - 10$.

(b) Note that the length of each cycle of Petersen graph is at least 5, and the number of vertices and edges are 10 and 15 respectively. Therefore, $3E = 45 > 40 = 5V - 10$ which implies that the Petersen graph is non-planar.

8. Let G be a simple graph with at least 11 vertices and let $\overline{G}$ be its complement. Prove that G and $\overline{G}$ cannot be both planar.

Ans:

Let V and E be the number of vertices and edges of G respectively. Then $\overline{G}$ has V vertices as well. Also, if $\overline{G}$ has E' edges, then $E + E' = \dfrac{V(V-1)}{2}$ since the union of E and E' is the complete graph.

Suppose that both G and $\overline{G}$ is planar, then we have $E \leq 3V - 6$ and $E' \leq 3V - 6$ (Corollary obtained from the Euler's formula and each face is bounded by at least 3 edges). If we add them up, we have $\dfrac{V(V-1)}{2} \leq 6V - 12$ which implies $V^2 - 13V + 24 \leq 0$ and so $8 < \dfrac{13 - \sqrt{73}}{2} \leq V \leq \dfrac{13 + \sqrt{73}}{2} < 11$. The result follows.

Remark: In fact, more techniques can be used to prove that the above statement is true for a simple graph with at least 9 vertices.

However, you can find a simple graph with 8 vertices such that both G and $\overline{G}$ are planar.